

The Schwinger model on S^1 : Hamiltonian formulation, vacuum and anomaly

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Abstract

We present a Hamiltonian formulation of the Schwinger model on the circle in Coulomb gauge as a semi-bounded self-adjoint operator which is invariant under the modular group $\mathcal{M} = \mathbb{Z}$ of large gauge transformations. There is a nontrivial action of \mathcal{M} on fermionic Fock space \mathcal{H}_0 and its vacuum which plays a role analogous to that of the spectral flow in the formalism involving the infinite Dirac sea. The formulation allows (i) a description of the anomaly and its relation to this group action, and (ii) an explicit identification of the interacting vacuum which arises after the destabilization of the non-interacting vacuum in \mathcal{H}_0 .

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1 Introduction

The Schwinger model is two dimensional quantum electrodynamics with massless fermions. The action functional is

$$S = \int -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}i\not{D}\psi \, dx \, dt, \quad \not{D} = \gamma^\mu(\partial_\mu - ie a_\mu) \quad (1.1)$$

describing the interaction of a Dirac spinor field ψ with an electromagnetic potential $a_\mu dx^\mu$ with associated electromagnetic field $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. The model was shown to be formally solvable by Schwinger 50 years ago in [11], and to possess the interesting property that although the classical field theory describes massless particles, the quantum field theory describes scalar bosons of mass $e/\sqrt{\pi}$. This feature is closely related to the presence of an *anomaly*, i.e. a symmetry of the classical theory which is not shared by the quantum theory (in this case chiral phase rotation $\psi \mapsto e^{i\gamma^5\theta}\psi$). A mathematical expression of the anomaly is that there is a current $j^{5,\mu}$ associated to the symmetry which is classically conserved (i.e. divergence free) but in the quantum theory satisfies

$$\partial_\mu j^{5,\mu} = -\frac{1}{\pi}E, \quad (1.2)$$

where $E = \dot{a} - \partial a_0$ is the electric field.

In this letter we consider the Schwinger model on the circle in Coulomb gauge, as in [9], with particular attention paid to the role of the *modular* group \mathcal{M} of “large” gauge transformations which are left unfixed by Coulomb gauge. But in contrast to the infinite Dirac sea formalism of [9] we use a positive energy representation (for the fermion field) which allows a formulation of the Hamiltonian as a self-adjoint operator in a mathematically precise way, via classical bosonization results from [8, 14]. The interest lies in the rigorous and transparent explanation of the anomaly, and the related appearance of the mass $e/\sqrt{\pi}$, in terms of a nontrivial action of \mathcal{M} on the non-interacting Fock vacuum - this latter action replaces the role played by the spectral flow in the infinite Dirac sea formalism for the anomaly given in [9].

In this section we introduce as basis for the discussion the expression for the second quantized Hamiltonian ((1.15)-(1.17)) which is derived from Schwinger gauge invariant regularization in [12]. As a starting point we take the Hamiltonian formulation of the classical Schwinger model. We work in $1+1$ dimensional space-time with coordinates (t, x) and metric $dt^2 - dx^2$, with $0 \leq x \leq L$ and periodic boundary conditions.

The dependent variables are a Dirac field $\psi(t, x) \in \mathbb{C}^2$ and an electromagnetic connection form $a_\mu dx^\mu = a_0 dt + a dx$. The gauge transformations act as

$$\psi \rightarrow e^{ig} \psi \quad \text{and} \quad a_\mu \rightarrow a_\mu + \partial_\mu g$$

where $g = g(t, x)$ is a sufficiently regular function *which is L periodic in x* . As in [9] we will work in the Coulomb gauge, in which the spatial component of the connection a depends only on time so that the expression for the electric field $E = \dot{a} - \partial a_0$ is in fact the decomposition into the longitudinal and transverse components: $E^{long} = -\partial a_0$ and $E^{tr} = \dot{a}$ respectively. The time component a_0 is integrated out via the Gauss law leading to the following classical Hamiltonian in the zero mass case:

$$\int_0^L \frac{1}{2e^2} \dot{a}^2 - \psi^\dagger (i\gamma^5 (\partial - ia)\psi) + \frac{1}{2} e^2 (\psi^\dagger \psi) (-\Delta)^{-1} * (\psi^\dagger \psi) dx. \quad (1.3)$$

Here $(-\Delta)^{-1}$ means the kernel of the operator $-\Delta = -\partial^2$ on $[0, L]$ with periodic boundary conditions, $*$ is convolution and $\partial = \partial_x$. Notice that the longitudinal component of the electric field has been integrated out leaving only the transverse component $E^{tr} = \dot{a}$. We use the following form of the gamma matrices:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad (1.4)$$

and we will use dots (resp. ∂) to indicate derivatives with respect to t (resp. x).

Remark 1 Notice that, in contrast to the case when space is the whole real line, the periodicity requirement means it is not possible to choose a gauge in which the spatial component of the connection a is actually zero, only spatially constant. However in this formulation there is a residual gauge invariance by the modular group

$$\mathcal{M} = \mathbb{Z} = \{g_N(x) = e^{2\pi i N x/L}\}_{N \in \mathbb{Z}}$$

of large gauge transformations. Notice as a first consequence a is now defined mod $2\pi/L$ so that it is now taking values in the circle $S^1 = \mathbb{R}/(2\pi/L)$ which is dual to the spatial domain \mathbb{R}/L . We shall see that a careful treatment of the residual component of the potential $a = a(t)$ and invariance under the group \mathcal{M} illuminates greatly the role of gauge invariance in producing the anomaly and the interacting vacuum.

The classical equations of motion associated to (1.3) are

$$i\dot{\psi} = -i\gamma^5 (\partial\psi - ia\psi) - a_0\psi$$

$$\dot{E}^{tr} = \frac{e^2}{L} \int_0^L \psi^\dagger \gamma^5 \psi dx, \quad \dot{a} = E^{tr}, \quad (1.5)$$

where a_0 is determined by the Gauss law constraint $-\Delta a_0 = -e^2 \psi^\dagger \psi = -e^2 j^0$. We will write $j^0 = \psi^\dagger \psi$ and $j^1 = \psi^\dagger \gamma^5 \psi$ for the currents and $Q = \int_0^L j^0 dx$, $Q^5 = \int_0^L j^1 dx$ for the corresponding charges. In order that the Gauss law admit a periodic solution it is necessary that $Q = 0$, so that *throughout it will be assumed that the total charge is zero*. In the classical theory both the electromagnetic current $j = j^\mu = \bar{\psi} \gamma^\mu \psi = (j^0, j^1)$ and the axial current $j^{5,\mu} = \bar{\psi} \gamma^\mu \gamma^5 \psi = (j^1, j^0)$ are conserved, but in the quantum theory only the first of these properties holds - the conservation law for j^5 is replaced by the anomaly equation (1.2), see §6. As a first intimation of the connection of (1.2) with mass generation notice that together with (1.5) it implies that the electric field satisfies $(\square + \frac{e^2}{\pi})E = 0$ in place of $\square E = 0$ - thus the anomalous right hand side of (1.2) generates a mass $e/\sqrt{\pi}$. However rather than start with (1.2) we will approach the problem through the Hamiltonian, and the reason for the anomaly will appear in §2 as a consequence of defining a regularized Hamiltonian which is invariant under the action of \mathcal{M} ; see also the comments at the end of section 6, in which (1.2) is finally derived.

To quantize the theory it is necessary to associate operators to the fields which satisfy the canonical relations:

$$\{\psi_\alpha(t, x), \psi_\beta^\dagger(t, y)\} = \delta_{\alpha\beta} \delta(x - y) \quad (1.6)$$

(other anti-commutators being zero), and

$$[E^{tr}, a] = [\dot{a}, a] = -\frac{ie^2}{L} \quad (1.7)$$

(other commutators being zero). For the latter we will use the Schrödinger representation in which a is represented by coordinate multiplication on $L^2([0, \frac{2\pi}{L}])$, while

$$E^{tr} = -\frac{ie^2}{L} \frac{d}{da}.$$

In the absence of interaction with any matter fields the electromagnetic field is described by the Hamiltonian $H_{em} = -\frac{e^2}{2L} \frac{d^2}{da^2}$ on $L^2([0, \frac{2\pi}{L}])$ with periodic boundary conditions - the large gauge transformations described in remark 1 are the reason that periodic boundary conditions are appropriate. We shall see below how these boundary conditions are modified in the presence of interactions with fermionic matter.

The relations (1.6) are interpreted in the positive energy representation by writing

$$\psi = \frac{1}{\sqrt{L}} \sum_{n \in \mathbb{Z}} (b_n u_n e^{ik_n x} + c_n^\dagger v_n e^{-ik_n x}), \quad k_n = \frac{2n\pi}{L} \quad (1.8)$$

with

$$\{b_n, b_{n'}^\dagger\} = \{c_n, c_{n'}^\dagger\} = \delta_{nn'} \quad (1.9)$$

(other anti-commutators being zero) and

$$\begin{aligned} u_n &= u^R \mathbb{1}_{\{n \geq 0\}} + u^L \mathbb{1}_{\{n < 0\}}, \\ v_n &= u^R \mathbb{1}_{\{n > 0\}} + u^L \mathbb{1}_{\{n \leq 0\}}. \end{aligned} \quad (1.10)$$

The $u^{L,R}$ are eigenvectors of γ^5 with $\gamma^5 u^R = u^R$ and $\gamma^5 u^L = -u^L$. The b_m^\dagger, b_m (resp. c_m^\dagger, c_m) are fermionic (resp. anti-fermionic) creation, annihilation operators acting on the zero charge fermionic Fock space \mathcal{H}_0 . The total Hilbert space for the theory can now be defined as

$$\mathcal{K} = \{\Psi = \Psi(a) \in \mathcal{H}_0 : \Psi \in L^2([0, \frac{2\pi}{L}]; \mathcal{H}_0)\}, \quad (1.11)$$

with norm defined by $\|\Psi\|_{\mathcal{K}}^2 = \int_0^{\frac{2\pi}{L}} \|\Psi\|^2 da$ where $\|\cdot\|$ is the Fock space norm. Recall the fermionic Fock space: there is a (non-interacting) vacuum Ω_0 and associated finite particle states

$$\Omega_{\mathbf{m}, \mathbf{n}} = \prod b_{m_i}^\dagger c_{n_j}^\dagger \Omega_0 \quad (1.12)$$

where $\mathbf{m} = \{m_i\}_{i=1}^M$ and $\mathbf{n} = \{n_j\}_{j=1}^N$ range over subsets of \mathbb{Z} of arbitrary finite size. (In (1.12) and similar formulae it will be assumed that products of creation/annihilation operators are ordered from left to right as the indices m_i or n_j increase. This fixes the overall sign.) Let \mathcal{F} be the linear span of all the $\Omega_{\mathbf{m}, \mathbf{n}}$, let $\mathcal{F}_0 \subset \mathcal{F}$ be the zero charge subspace in which there are equal numbers of fermions and anti-fermions, i.e. $M = N$. The zero charge Fock space \mathcal{H}_0 is the completion of \mathcal{F}_0 in the Fock space norm $\|\cdot\|$, and the vectors in (1.12) constitute an orthonormal basis. There is a self-adjoint operator which extends the operator given on \mathcal{F}_0 by

$$Q^5 = \sum_{n \geq 0} b_n^\dagger b_n - \sum_{n < 0} b_n^\dagger b_n - \sum_{n > 0} c_n^\dagger c_n + \sum_{n \leq 0} c_n^\dagger c_n.$$

which will also be denoted Q^5 ; it will be referred to as the *axial* (or *chiral*) charge operator. Define $\mathcal{F}_0^P \subset \mathcal{F}_0 = \{\text{Ker}(Q^5 - 2P)\} \cap \mathcal{F}_0$. The corresponding completions are denoted \mathcal{H}_0^P , and are the orthogonal eigenspaces arising in the spectral decomposition of Q^5 .

We define the *unexcited* states Ω_P in the case when $M = N = P \in \mathbb{Z}^+$ as follows. The case $P = 0$ corresponds to the vacuum Ω_0 . For $P \geq 0$ let $m_i = n_i = i$ for $0 \leq i \leq P$, and define the unexcited state

$$\Omega_{P+1} = \prod_{i=0}^P \prod_{j=0}^P b_{m_i}^\dagger c_{-n_j}^\dagger \Omega_0 \quad (P \in \mathbb{Z}^+); \quad (1.13)$$

for $P < 0$ and $M = N = -P$ let $m_i = n_i = -i$ for $0 < i \leq -P$, and define the unexcited state as

$$\Omega_P = \prod_{i=-1}^P \prod_{j=-1}^P b_{m_i}^\dagger c_{-n_j}^\dagger \Omega_0 \quad (P \in \mathbb{Z}^- - \{0\}). \quad (1.14)$$

The representation of (1.6) used in (1.8) is a positive energy representation in that the free Dirac Hamiltonian $H_D^0 = \sum_{m \in \mathbb{Z}} |k_m| (b_m^\dagger b_m + c_m^\dagger c_m) \geq 0$.

In the process of quantizing it is necessary to define carefully what is meant mathematically by the various formal expressions for bilinear quantities such as those for the axial charge and the Hamiltonian itself, which involve products of what are at best operator valued distributions. As emphasized in [11] the definition needs to be chosen carefully to ensure gauge invariance is maintained. The least intrusive way of doing this seems to be by Schwinger regularization (point-splitting), and the relevant computations are presented in some notes available online ([12]). The endpoint of this is the following formula for the regularized Hamiltonian: $H = H_0 + : H_{coul} :$, where

$$H_0 = -\frac{e^2}{2L} \frac{d^2}{da^2} + \sum_{m \in \mathbb{Z}} |k_m| (b_m^\dagger b_m + c_m^\dagger c_m) - \frac{a^2 L}{2\pi} - a Q^{5,reg}, \quad (1.15)$$

and

$$H_{coul} = \frac{e^2 L}{2} \sum_{m \neq 0} \frac{1}{k_m^2} j^0(-m) j^0(m) \quad (1.16)$$

is the Coulomb energy, written in terms of the Fourier modes of the current operator

$$j^0 = \sum j^0(m) e^{ik_m x}.$$

In (1.15) the symbol $Q^{5,reg}$ indicates the regularized axial charge operator given by:

$$Q^{5,reg} = \sum_{n \geq 0} b_n^\dagger b_n - \sum_{n < 0} b_n^\dagger b_n - \sum_{n > 0} c_n^\dagger c_n + \sum_{n \leq 0} c_n^\dagger c_n - \frac{aL}{\pi} - 1. \quad (1.17)$$

This expression is also derived from Schwinger regularization in [12], where it is also shown that the corresponding expression for the regularized ordinary charge is in fact unchanged, i.e.

$$Q = Q^{reg} = \sum_{n \in \mathbb{Z}} (b_n^\dagger b_n - c_n^\dagger c_n).$$

Formulae closely related to (1.15)-(1.17), but in the infinite Dirac sea context, can be found in [9, §3], where they are derived using a gauge invariant heat kernel regularization to handle the arbitrarily unbounded negative energies which arise in that formulation.

The aim in this letter is to recall how classical bosonization results can be used to make sense of the above expression for H as a self-adjoint operator on the Hilbert space \mathcal{K} , and thence to clarify the vacuum structure and the anomaly in a mathematically precise way. These clarifications hinge upon an understanding of the gauge transformations, so we first discuss the action of the group \mathcal{M} of large gauge transformations. This leads to the correct boundary condition (2.29) which is required to complete the mathematical formulation of the Schwinger model in terms of the Hamiltonian (1.15)-(1.17).

2 Action of \mathcal{M} and twisted periodicity

We define a unitary action of the group $\mathcal{M} = \mathbb{Z} = \{g_N(x) = e^{2\pi i N x/L}\}_{N \in \mathbb{Z}}$ of large gauge transformations on \mathcal{H}_0 . The formulae are best motivated by comparison with the natural expressions in the infinite Dirac sea - see [9, 12]. There is a unitary operator Γ , corresponding to the generator g_1 , whose action on the non-interacting vacuum state is

$$\Gamma \Omega_0 = \Omega_{-1} = b_{-1}^\dagger c_1^\dagger \Omega_0. \quad (2.18)$$

The action on Fock space is then determined by specifying the action on the set of creation and annihilation operators, on which it acts as a modified shift operator:

$$\begin{aligned} b_n &\rightarrow \Gamma b_n \Gamma^{-1} = b_{n-1}, \quad n \neq 0, & b_0 &\rightarrow \Gamma b_0 \Gamma^{-1} = c_1^\dagger \\ c_n &\rightarrow \Gamma c_n \Gamma^{-1} = c_{n+1}, \quad n \neq 0, & c_0 &\rightarrow \Gamma c_0 \Gamma^{-1} = b_{-1}^\dagger \end{aligned} \quad (2.19)$$

with corresponding relations for the adjoints:

$$\begin{aligned} b_n^\dagger \rightarrow \Gamma b_n^\dagger \Gamma^{-1} &= b_{n-1}^\dagger, \quad n \neq 0, & b_0^\dagger \rightarrow \Gamma b_0^\dagger \Gamma^{-1} &= c_1 \\ c_n^\dagger \rightarrow \Gamma c_n^\dagger \Gamma^{-1} &= c_{n+1}^\dagger, \quad n \neq 0, & c_0^\dagger \rightarrow \Gamma c_0^\dagger \Gamma^{-1} &= b_{-1} \end{aligned} \quad (2.20)$$

Lemma 2 *The formulae (2.18)-(2.20) determine an action of $\mathcal{M} = \mathbb{Z}$ on \mathcal{H}_0 generated by Γ , with the property that $\Gamma \Omega_P = \Omega_{P-1}$ for all P . Similarly there is a corresponding modified shift action for the inverse Γ^{-1} with (2.18)-(2.20) inverted, so that in particular $\Gamma^{-1} \cdot \Omega_0 = b_0^\dagger c_0^\dagger \Omega_0 = \Omega_1$ and more generally $\Gamma^{-1} \cdot \Omega_P = \Omega_{P+1}$.*

Proof This is a consequence of the fact that Γ acts as a bijection on the set of orthonormal basis vectors $\{\Omega_{\mathbf{m},\mathbf{n}}\}$ labelled by pairs of finite subsets of \mathbb{Z} : in fact $\Gamma \Omega_{\mathbf{m},\mathbf{n}} = \iota_{\mathbf{m},\mathbf{n}} \Omega_{\mathbf{m}',\mathbf{n}'}$ where $\iota_{\mathbf{m},\mathbf{n}} \in \{\pm 1\}$ is an unimportant overall sign and the subsets \mathbf{m}', \mathbf{n}' are given by

$$\mathbf{m}' = \cup_{m \in \mathbf{m}, m \neq 0} \{m-1\} \quad (0 \in \mathbf{n}) \quad (2.21)$$

$$= \cup_{m \in \mathbf{m}, m \neq 0} \{m-1\} \cup \{-1\} \quad (0 \notin \mathbf{n}) \quad (2.22)$$

$$\mathbf{n}' = \cup_{n \in \mathbf{n}, n \neq 0} \{n+1\} \quad (0 \in \mathbf{m}) \quad (2.23)$$

$$= \cup_{n \in \mathbf{n}, n \neq 0} \{n+1\} \cup \{+1\} \quad (0 \notin \mathbf{m}). \quad (2.24)$$

This is clearly a bijection on pairs of subsets of \mathbb{Z} of arbitrary finite size, whose inverse is of the same form: $\Gamma^{-1} \Omega_{\mathbf{m},\mathbf{n}} = \iota_{\mathbf{m},\mathbf{n}} \Omega_{\mathbf{m}',\mathbf{n}'}$ where

$$\mathbf{m}' = \cup_{m \in \mathbf{m}, m \neq -1} \{m+1\} \quad (1 \in \mathbf{n}) \quad (2.25)$$

$$= \cup_{m \in \mathbf{m}, m \neq -1} \{m+1\} \cup \{0\} \quad (1 \notin \mathbf{n}) \quad (2.26)$$

$$\mathbf{n}' = \cup_{n \in \mathbf{n}, n \neq 1} \{n-1\} \quad (-1 \in \mathbf{m}) \quad (2.27)$$

$$= \cup_{n \in \mathbf{n}, n \neq 1} \{n-1\} \cup \{0\} \quad (-1 \notin \mathbf{m}), \quad (2.28)$$

as can readily be verified. In the same way this corresponds to Γ^{-1} , formulae for which appear in [12, §3.1]. All together this determines a unitary transformation

$$\Gamma \sum \psi_{\mathbf{m},\mathbf{n}} \Omega_{\mathbf{m},\mathbf{n}} = \sum \iota_{\mathbf{m},\mathbf{n}} \psi_{\mathbf{m},\mathbf{n}} \Omega_{\mathbf{m}',\mathbf{n}'}$$

of \mathcal{H}_0 which induces (2.18)-(2.20). \square

The transformation Γ commutes with Q and so preserves \mathcal{H}_0 , but it does not commute with Q^5 : for example $b_3^\dagger c_2^\dagger b_0^\dagger c_1^\dagger \Omega_0$ is mapped into $b_2^\dagger c_3^\dagger c_2^\dagger b_{-1}^\dagger \Omega_0$, with the eigenvalue of Q^5 reducing by 2. Formally $Q^5 \Gamma^{-1} = \Gamma^{-1} (Q^5 - 2)$ on \mathcal{F}_0 . The interpretation of all these formulae is that *large gauge transformations can create and annihilate fermion/anti-fermion pairs* in a way which seems naively to change the axial charge: an *anomaly*. Nevertheless we have:

Lemma 3 *The Schwinger regularizations of the axial charge (1.17), and of the Hamiltonian (1.15)-(1.16), are unchanged by the action of \mathcal{M} .*

Proof This is straightfoward to check, see [12, §3.4]. \square

Now the gauge transformation g_1 acts on the connection as $a \rightarrow a + \frac{2\pi}{L}$, and hence the requirement of gauge invariance means that we should regard the Hamiltonian H as an unbounded operator defined on \mathcal{K} with the following boundary conditions of *twisted periodicity*:

$$\Psi\left(\frac{2\pi}{L}\right) = \Gamma^{-1} \Psi(0) \quad \text{and} \quad \Psi'\left(\frac{2\pi}{L}\right) = \Gamma^{-1} \Psi'(0). \quad (2.29)$$

(writing prime for $\frac{d}{da}$). A suitable dense domain for the Hamiltonian is \mathcal{D} , the space of smooth functions taking values in \mathcal{F}_0 which satisfy this twisted periodicity condition, i.e. the restriction to $[0, \frac{2\pi}{L}]$ of the smooth \mathcal{F}_0 -valued functions which satisfy $\Psi(a + \frac{2\pi}{L}) = \Gamma^{-1} \Psi(a)$ for all $a \in \mathbb{R}$.

Lemma 4 $\mathcal{D} \subset \mathcal{K}$ is dense in the norm $\|\cdot\|_{\mathcal{K}}$ on \mathcal{K} . The integration by parts formula $\langle \Psi', \Phi \rangle_{\mathcal{K}} = -\langle \Psi, \Phi' \rangle_{\mathcal{K}}$ holds for Ψ, Φ in \mathcal{D} .

Proof It suffices to first approximate $\Psi \in \mathcal{K}$ by simple functions $\sum \mathbb{1}_{I_j}(a)f_j$ where $f_j \in \mathcal{F}_0$ and I_j are measurable sets contained in a closed sub-interval of $(0, \frac{2\pi}{L})$. Then approximate the characteristic functions $\mathbb{1}_{I_j}(a)$ by smooth functions compactly supported in $(0, \frac{2\pi}{L})$. The twisted periodicity condition is then trivially satisfied. The integration by parts formula holds since $\mathbf{\Gamma}$ is unitary on \mathcal{H}_0 . \square

Remark 5 The Fock vacuum Ω_0 , thought of as an element of \mathcal{K} which is independent of a , does not satisfy (2.29) and is not gauge invariant. It follows that the interacting (or physical) vacuum cannot be proportional to Ω_0 , or indeed any of the unexcited states Ω_P , since \mathcal{M} maps these states into one another, thus destabilizing the Fock vacuum. We shall see in §4 that the physical vacuum is a linear combination of states of the form $f_P(a)\Omega_P$.

Remark 6 If Ψ satisfies (2.29) then $\tilde{\Psi} = e^{i\theta Q^{5,reg}} \Psi$ satisfies $\Psi(\frac{2\pi}{L}) = e^{-2i\theta} \mathbf{\Gamma}^{-1} \Psi(0)$, corresponding to a phase change in the definition of $\mathbf{\Gamma}$ (which is clearly allowed by the above discussion). The parameter is called the θ parameter, and this transformation shows that it does not give any new physics, but rather corresponds to the choice of an equivalent representation for E^{tr} - see [9, §2 and §6].

3 Bosonization

From [8], and earlier references therein, it is known that associated to a free massless fermionic field is a free real scalar field Φ with conjugate momentum Π , given at fixed time by:

$$\begin{aligned} \Phi(x) &= \sum \Phi_m e^{ik_m x}, & \Phi_m^\dagger &= \Phi_{-m} \\ \Pi(x) &= \sum \Pi_m e^{ik_m x}, & \Pi_m^\dagger &= \Pi_{-m} \end{aligned} \quad (3.1)$$

where $k_m = 2m\pi/L$ for $m \in \mathbb{Z}$, and with $[\Pi_{-m}, \Phi_{m'}] = -\frac{i}{L} \delta_{mm'}$ (all other commutators being zero). This implies the relation $[\Pi(x'), \Phi(x)] = -\frac{i}{L} \sum e^{ik_m(x-x')} = -i\delta(x-x')$. The relevant representation of these commutation relations is related to the particle structure and as such will be determined later. The final conclusion will be the identification, after normal ordering and a shift of the vacuum energy, of the Hamiltonian for the Schwinger model with the Hamiltonian:

$$H_S = \frac{1}{2} \int_0^L \left(\Pi(x)^2 + \partial \Phi(x)^2 + \frac{e^2}{\pi} \Phi(x)^2 \right) dx, \quad (3.2)$$

for a real scalar field with mass $e/\sqrt{\pi}$. This is to be expected from Schwinger's work [11].

The relation of Φ, Π to the fermionic and electromagnetic fields of the Schwinger model is given by the formulae:

$$\begin{aligned} \Phi_m &= -\frac{\sqrt{\pi}}{ik_m} j^0(m), & \Pi_m &= \sqrt{\pi} j^1(m), & (m \neq 0) \\ \Phi_0 &= \frac{\sqrt{\pi}}{e^2} E^{tr}, & \Pi_0 &= \frac{\sqrt{\pi}}{L} Q^{5,reg} = \frac{\sqrt{\pi}}{L} (Q^5 - \frac{aL}{\pi} - 1). \end{aligned} \quad (3.3)$$

Here $j^\mu(m)$ are the fourier modes of the current operators $j^\mu = \bar{\psi} \gamma^\mu \psi$:

$$j^0 = \sum j^0(m) e^{ik_m x}, \quad j^1 = \sum j^1(m) e^{ik_m x}. \quad (3.4)$$

The $j^\mu(m)$ may be obtained from the fermionic creation/annihilation operators from the formulae (for $m \in \mathbb{N}$):

$$j^1(m) = \frac{1}{L} [\varrho^R(-m) - \varrho^L(-m)], \quad j^1(-m) = \frac{1}{L} [\varrho^R(m) - \varrho^L(m)], \quad (3.5)$$

$$j^0(m) = \frac{1}{L} [\varrho^R(-m) + \varrho^L(-m)], \quad j^0(-m) = \frac{1}{L} [\varrho^R(m) + \varrho^L(m)], \quad (3.6)$$

where

$$\varrho^R(m) = - \sum_{k>m} c_k^\dagger c_{k-m} + \sum_{0<k\leq m} b_{-k+m}^\dagger c_k^\dagger + \sum_{k\geq 0} b_{k+m}^\dagger b_k, \quad (3.7)$$

$$\varrho^R(-m) = - \sum_{k>m} c_k^\dagger c_{k-m} + \sum_{0<k\leq m} c_k b_{-k+m} + \sum_{k\geq 0} b_k^\dagger b_{k+m}, \quad (3.8)$$

$$\varrho^L(m) = \sum_{k<-m} b_{k+m}^\dagger b_k + \sum_{-m\leq k<0} c_{-k-m} b_k - \sum_{k\leq 0} c_k^\dagger c_{k-m}, \quad (3.9)$$

$$\varrho^L(-m) = \sum_{k<-m} b_k^\dagger b_{k+m} + \sum_{-m\leq k<0} b_k^\dagger c_{-k-m}^\dagger - \sum_{k\leq 0} c_{k-m}^\dagger c_k. \quad (3.10)$$

The expressions (3.7)-(3.10) arise by considering the fourier expansion of the density operators $\rho^{R,L}(x) = \frac{1}{2}\psi^\dagger(1 \pm \gamma^5)\psi(x)$. They are precisely the quantities appearing in [8, (3.1)-(3.4)] except for some conventions (ordering and the sign of the integer index on the c_k operators has been reversed.) Notice the following two important features:

- Gauge invariance: $\mathbf{\Gamma}\rho^{R,L}(m)\mathbf{\Gamma}^{-1} = \rho^{R,L}(m)$;
- on the finite particle subspace \mathcal{F}_0 the expressions (3.7)-(3.10) reduce to finite sums, and direct computation yields:

$$[\varrho^R(-m'), \varrho^R(m)] = [\varrho^L(m'), \varrho^L(-m)] = m\delta_{mm'}, \quad (3.11)$$

for positive integral m, m' , other commutators being zero.

The relations (3.11) imply the commutation relation $[\Pi_{-m}, \Phi_{m'}] = -\frac{i}{L}\delta_{mm'}$ required to ensure that the field defined as in (3.1) is a canonical scalar field. The commutation relations (3.11) suggest that for $m \in \mathbb{N}$ the $m^{-\frac{1}{2}}\varrho^R(-m), m^{-\frac{1}{2}}\varrho^L(m)$ (resp. $m^{-\frac{1}{2}}\varrho^R(m), m^{-\frac{1}{2}}\varrho^L(-m)$) represent annihilation (resp. creation) operators. In addition notice that in (3.7) and (3.10) (resp. (3.8) and (3.9)) the first and third terms merely shift the momentum of fermions already present, while the middle terms create (resp. annihilate) two fermions of opposite chirality, so that $Q^5 = 2P$ is unchanged. This motivates the following result:

Proposition 7 ([14], §4) *The subspaces $\mathcal{H}_0^P \subset \mathcal{H}_0$ are irreducible cyclic subspaces for the algebra of operators generated by $\{\varrho^R(m'), \varrho^L(m)\}_{m,m' \in \mathbb{Z} - \{0\}}$, giving rise to a Fock representation of the canonical commutation relations with cyclic vector Ω_P which verifies $\varrho^R(-m)\Omega_P = 0 = \varrho^L(+m)\Omega_P$ for $m \in \mathbb{N}$.*

In [8] it is pointed out that the fermionic kinetic energy operator

$$H_D^0 = \sum_{m \in \mathbb{Z}} |k_m| (b_m^\dagger b_m + c_m^\dagger c_m)$$

has the same commutation relations with the operators $\{\varrho^R(m'), \varrho^L(m)\}_{m,m' \in \mathbb{Z} - \{0\}}$ as the nonnegative operator

$$T = \frac{2\pi}{L} \sum_{m \in \mathbb{N}} (\rho^R(m)\rho^R(-m) + \rho^L(-m)\rho^L(m)) \geq 0 \quad (3.12)$$

(on each finite particle subspace \mathcal{F}_0^P). From this it follows from the above proposition that on \mathcal{F}_0^P

$$H_D^0 = T + \langle P | H_D^0 | P \rangle = T + \frac{\pi}{2L} Q^5 (Q^5 - 2) \quad (3.13)$$

(Kronig's identity). In fact the identity (3.13) extends to an equality between self-adjoint operators on \mathcal{H}_0 , see [14, §5]. Combining with (1.15) we obtain the following bosonized formula for (1.15):

$$H_0 = -\frac{e^2}{2L} \frac{d^2}{da^2} + \frac{\pi}{2L} (Q^{5,reg})^2 + T. \quad (3.14)$$

4 The vacuum for H_0

Since integration by parts is allowed by lemma 4 and $T \geq 0$ as an operator inequality:

$$\langle \Psi, H_0 \Psi \rangle_{\mathcal{K}} \geq \frac{e^2}{2L} \|\Psi'\|_{\mathcal{K}}^2 + \frac{\pi}{2L} \|Q^{5,reg} \Psi\|_{\mathcal{K}}^2, \quad (4.1)$$

on \mathcal{D} . Next, using the orthogonal decomposition $\mathcal{H}_0 = \oplus \mathcal{H}_0^P$, we can write $\Psi = \sum \Psi_P$ with $\Psi_P(a) \in \mathcal{H}_0^P \cap \mathcal{F}_0 \forall a$ and since $Q^{5,reg} = 2P - \frac{aL}{\pi} - 1$ on \mathcal{H}_0^P we get

$$\langle \Psi, H_0 \Psi \rangle_{\mathcal{K}} \geq \sum_P \left[\frac{e^2}{2L} \|\Psi_P'\|_{\mathcal{K}}^2 + \frac{\pi}{2L} \|(2P - \frac{aL}{\pi} - 1) \Psi_P\|_{\mathcal{K}}^2 \right], \quad (4.2)$$

$$\geq \sum_P \frac{e}{2\sqrt{\pi}} \|\Psi_P\|_{\mathcal{K}}^2 = E_0 \|\Psi\|_{\mathcal{K}}^2, \quad (4.3)$$

where $E_0 = \frac{e}{2\sqrt{\pi}}$, using the standard lower bound for the oscillator which follows from the commutation relation $[\frac{d}{da}, a] = 1$. Thus $H_0 \geq E_0$ on \mathcal{D} .

We will now show that this lower bound E_0 is realized on states of the form

$$\Psi_0(a) = \sum f_P(a) \Omega_P. \quad (4.4)$$

The twisted periodic boundary condition (2.29) translates into the requirements

$$f_{P+1}(\frac{2\pi}{L}) = f_P(0), \quad \text{and} \quad f'_{P+1}(\frac{2\pi}{L}) = f'_P(0) \quad (4.5)$$

for the sequence of functions. We aim to solve $H_S \Psi_0 = E_0 \Psi_0$ under these conditions. Defining

$$f(\tilde{a}) = f_P(\tilde{a} + \frac{2\pi P}{L} - \frac{\pi}{L}) \quad \text{for} \quad \tilde{a} \in I_P = [-\frac{2\pi}{L}(P - \frac{1}{2}), -\frac{2\pi}{L}(P - \frac{3}{2})]$$

gives a function on the real line, and the eigenvalue equation $H_S \Psi_0 = E_0 \Psi_0$ is equivalent to the oscillator Schrodinger equation

$$-\frac{e^2}{2L} \frac{d^2 f}{d\tilde{a}^2} + \frac{L}{2\pi} \tilde{a}^2 f = E_0 f \quad (4.6)$$

which has a solution for $E_0 = e/(2\sqrt{\pi})$ proportional to $e^{\frac{-L\tilde{a}^2}{2\sqrt{\pi}e}}$. Normalizing we define

$$f(\tilde{a}) = \frac{L^{\frac{1}{4}}}{\pi^{\frac{3}{8}} e^{\frac{1}{4}}} e^{\frac{-L\tilde{a}^2}{2\sqrt{\pi}e}}$$

and the vacuum state for H_0 is

$$\Psi_0(a) = \sum_{P \in \mathbb{Z}} f(a - \frac{2\pi}{L}(P - \frac{1}{2})) \Omega_P, \quad (4.7)$$

with normalization $\|\Psi_0\|_{\mathcal{K}} = 1$. This recovers one of the results of [9] but transferred to the positive energy representation.

5 The coulomb interaction

Next consider the Coulomb interaction, which is formally $\frac{1}{2}e^2 L \sum_{m \neq 0} j^0(-m) k_m^{-2} j^0(m)$. In order to obtain a densely defined operator it is necessary to subtract off the expectation with respect to the vacuum Ψ_0 in (4.7). Thus we consider the quadratic form

$$\frac{1}{2}e^2 L \sum_{m \neq 0} \frac{1}{k_m^2} (\|j^0(m) \Psi\|_{\mathcal{K}}^2 - \|j^0(m) \Psi_0\|_{\mathcal{K}}^2).$$

Noting that $\|j^0(m)\Omega_P\|^2 = |m|$ for each P it is easy to check from the exact expression (4.7) that this subtraction is equivalent to normal ordering with respect to the bosonic algebra described in proposition 7, and the corresponding operator is

$$: H_{coul} := \frac{1}{2}e^2L \sum_{m \neq 0} \frac{1}{k_m^2} : j^0(-m)j^0(m) : \quad (5.1)$$

with $: j^0(-m)j^0(m) :$ given by

$$\frac{1}{L^2}(\rho^R(m)\rho^R(-m) + \rho^L(m)\rho^R(-m) + \rho^R(m)\rho^L(-m) + \rho^L(-m)\rho^L(m)) \quad (5.2)$$

for $m \in \mathbb{N}$, and

$$\frac{1}{L^2}(\rho^R(-m)\rho^R(m) + \rho^L(m)\rho^R(-m) + \rho^R(m)\rho^L(-m) + \rho^L(m)\rho^L(-m)) \quad (5.3)$$

for $-m \in \mathbb{N}$. These two formulae can be combined into the formula

$$: j^0(-m)j^0(m) : = \frac{1}{L^2}(\rho^R(m)\rho^R(-m) + \rho^L(m)\rho^R(-m) + \rho^R(m)\rho^L(-m) + \rho^L(m)\rho^L(-m) - |m|) .$$

The fact that this operator is densely defined is now straightforward since up to a factor $\sqrt{|m|}$ the $\rho^{L,R}$ are just creation/annihilation operators, and the above expressions are normal ordered and are densely defined and essentially self-adjoint on the domain \mathcal{F}_0 . (See [2, Theorem 1] for a dense domain for the case of massive fermions when bosonization is not available in the simple form used here.)

Referring to (3.3) it is apparent that $: H_{coul} := \frac{e^2L}{2\pi} \sum_{m \neq 0} : \Phi_m^\dagger \Phi_m : ,$ and it is straightforward to check that formally

$$H' = T + : H_{coul} := \frac{L}{2} \sum_{m \neq 0} \left((k_m^2 + \frac{e^2}{\pi}) \Phi_m^\dagger \Phi_m + \Pi_m^\dagger \Pi_m \right) .$$

Note that T is normal ordered by definition, see (3.12). However Φ is not yet in the right representation: to complete the identification of the full Hamiltonian in (3.14)-(1.17) with (3.2), the Hamiltonian of a mass $\mu = e/\sqrt{\pi}$ scalar field, it is sufficient to obtain a unitary Bogoliubov transformation which diagonalizes $T + : H_{coul} :$ and puts the fields into the appropriate positive energy representation. Following [8, 14] this is done by means of a Bogoliubov transformation as follows: let $V_m = k_m^{-2}$ and $\lambda = \frac{1}{2}e^2$, then defining the real-valued even function $\zeta(m)$ by

$$\frac{\lambda V_m}{\lambda V_m + \pi} = -\tanh 2\zeta(m)$$

the self-adjoint operator

$$Z = \frac{2\pi i}{L} \sum_{m \neq 0} \frac{\zeta(m)}{k_m} \rho^R(m) \rho^L(-m)$$

generates a unitary transformation $\mathbf{U} = e^{iZ}$ with the property that (putting transformed operators into boldface):

$$\begin{aligned} \mathbf{H}' = \mathbf{U} H' \mathbf{U}^{-1} &= \frac{2\pi}{L} \sum_{m \in \mathbb{N}} \left(1 + \frac{2\lambda V_m}{\pi} \right)^{\frac{1}{2}} (\rho^R(m)\rho^R(-m) + \rho^L(-m)\rho^L(m)) \\ &\quad + \frac{2\pi}{L} \sum_{m \in \mathbb{N}} m \left[\left(1 + \frac{2\lambda V_m}{\pi} \right)^{\frac{1}{2}} - \frac{\lambda V_m}{\pi} - 1 \right] . \end{aligned} \quad (5.4)$$

For $m \in \mathbb{N}$ define

$$A_m^\dagger = -im^{-\frac{1}{2}}\rho^R(m), \quad A_m = im^{-\frac{1}{2}}\rho^R(-m), \quad (5.5)$$

and

$$A_{-m}^\dagger = im^{-\frac{1}{2}}\rho^L(-m), \quad A_{-m} = -im^{-\frac{1}{2}}\rho^L(m), \quad (5.6)$$

so that $[A_m, A_{m'}^\dagger] = \delta_{mm'}$ for a non-zero integral m, m' . In terms of these operators

$$\mathbf{H}' = \sum_{m \in \mathbb{N}} \left(k_m^2 + \frac{e^2}{\pi} \right)^{\frac{1}{2}} \left[A_m^\dagger A_m + A_{-m}^\dagger A_{-m} \right] + C_0 \quad (5.7)$$

where $C_0 \in (-\infty, 0)$ is the second line of (5.4).

This allows an identification of the interacting vacuum for the full Hamiltonian as follows. The Hamiltonian is given by

$$\mathbf{H} = \mathbf{U} \mathbf{H}' \mathbf{U}^{-1} = -\frac{e^2}{2L} \frac{d^2}{da^2} + \frac{\pi}{2L} (Q^{5,reg})^2 + \sum_{m \in \mathbb{N}} \left(k_m^2 + \frac{e^2}{\pi} \right)^{\frac{1}{2}} \left[A_m^\dagger A_m + A_{-m}^\dagger A_{-m} \right] + C_0. \quad (5.8)$$

We use boldface to distinguish the dressed forms of the various states, thus defining $\mathbf{\Omega}_P = \mathbf{U}^{-1} \Omega_P$ we see that the vacuum energy is $E_0 + C_0 < E_0$ with corresponding eigenfunction

$$\Psi_0(a) = \sum_{P \in \mathbb{Z}} f\left(a - \frac{2\pi P}{L}\right) \mathbf{\Omega}_P, \quad (5.9)$$

The effect of the Coulomb term on the vacuum is to shift the energy down by a finite amount $C_0 < 0$ and to map the non-interacting unexcited states Ω_P to their dressed versions $\mathbf{\Omega}_P$.

The fourier components of the dressed scalar field are given by

$$\Phi_m = \frac{1}{\sqrt{2\omega_m L}} (A_m + A_{-m}^\dagger), \quad (5.10)$$

$$\Pi_m = -i \sqrt{\frac{\omega_m}{2L}} (A_m - A_{-m}^\dagger), \quad (5.11)$$

with $\omega_m = \sqrt{k_m^2 + \frac{e^2}{\pi}}$ for $m \neq 0$, in terms of which

$$\begin{aligned} \mathbf{H} &= \mathbf{U} \mathbf{H}' \mathbf{U}^{-1} = -\frac{e^2}{2L} \frac{d^2}{da^2} + \frac{\pi}{2L} (Q^{5,reg})^2 + \frac{L}{2} \sum_{m \neq 0} : (\Pi_m^\dagger \Pi_m + (k_m^2 + \frac{e^2}{\pi}) \Phi_m^\dagger \Phi_m) : + C_0 \\ &= \frac{L}{2} \sum_{m \in \mathbb{Z}} : (\Pi_m^\dagger \Pi_m + (k_m^2 + \frac{e^2}{\pi}) \Phi_m^\dagger \Phi_m) : + C_0. \end{aligned} \quad (5.12)$$

(This formula is the normal ordered version of (3.2). The action of the dressing transformation \mathbf{U} is trivial on the $m = 0$ components, so that $\mathbf{\Phi}_0 = \Phi_0 = \frac{\sqrt{\pi}}{e^2} E^{tr}$, and $\mathbf{\Pi}_0 = \Pi_0 = \frac{\sqrt{\pi}}{L} Q^{5,reg}$.) The existence of a self-adjoint extension is now straightforward but nevertheless it is interesting to see the role of twisted periodicity in ensuring that \mathcal{D} is a domain of essential self-adjointness in the proof of the following theorem.

Theorem 8 *The symmetric operator \mathbf{H} is essentially self-adjoint on $\mathcal{D} \subset \mathcal{K}$ (with self-adjoint extension also written \mathbf{H}).*

Proof Since \mathbf{H} is bounded below it is sufficient to show that $\mathbf{H} + \lambda_0$ has dense range for large $\lambda_0 > 0$ by [10, theorem X.26]. Consider the orthonormal set of vectors in \mathcal{H}_0^P of the form

$$\Omega_P^{\mathbf{n}} = \text{const.} \prod_{m \neq 0} (A_m^\dagger)^{n_m} \Omega_P \quad (5.13)$$

labelled by $\mathbf{n} = \{n_m\}_{m \in \mathbb{Z} - \{0\}}$, with $n_m \in \mathbb{N} \cup \{0\}$ and only a finite number of the n_m nonzero. Since $\mathbf{\Gamma} \rho^{R,L}(m) \mathbf{\Gamma}^{-1} = \rho^{R,L}(m)$ it follows from lemma 2 that

$$\mathbf{\Gamma} \Omega_P^{\mathbf{n}} = \Omega_{P-1}^{\mathbf{n}}. \quad (5.14)$$

(Strictly speaking to achieve this it is necessary specify that the normalization constants in the definition of $\Omega_P^{\mathbf{n}}$ above are chosen independent of P , i.e. without any additional P dependent phase factors.) Now

linear combinations of functions of the orthonormal set $e^{ilLa}\Omega_{P_0}^{\mathbf{n}}$ span a dense set in \mathcal{K} and so as first stage we want to solve $(\mathbf{H} + \lambda_0)\Psi = e^{ilLa}\Omega_{P_0}^{\mathbf{n}}$ for $\Psi \in \mathcal{D}$. The boundary condition means that the solution must involve all values of P - as in (4.7) we obtain a solution of the form $\Psi = \sum_P f_P(a)\Omega_P^{\mathbf{n}}$. We can generate such a solution as follows: write $\mathbb{R} = \cup I_P$ where $I_P = [-\frac{2\pi}{L}(P - \frac{1}{2}), -\frac{2\pi}{L}(P - \frac{3}{2})]$ and define $f_P(a) = f(a - \frac{2\pi}{L}(P - \frac{1}{2}))$, $0 \leq a \leq \frac{2\pi}{L}$, $P \in \mathbb{Z}$, where $f(\tilde{a})$ is a function on \mathbb{R} which solves

$$-\frac{e^2}{2L} \frac{d^2 f}{d\tilde{a}^2} + \frac{L}{2\pi} \tilde{a}^2 f + (E_{\mathbf{n}}^{ex} + \lambda_0 + C_0)f = e^{ilL\tilde{a} - il\pi} \mathbf{1}_{I_{P_0}}(\tilde{a}),$$

and $E_{\mathbf{n}}^{ex} = \sum \omega_m n_m$ is the bosonic excitation energy. (The twisted boundary conditions (2.29) imply the matching conditions for f, f' exactly as in (4.5)). For $\lambda_0 + C_0 \geq 0$ it follows as in (4.3) that $\|f\|_{L^2(\mathbb{R})} \leq \text{const.} = |I_{P_0}|/E_0 = 2\pi/(LE_0)$. This is independent of P, P_0, l, \mathbf{n} , and so since the $\Omega_P^{\mathbf{n}}$ are of unit length in Fock space and orthogonal for different P we get $\|\Psi\|_{\mathcal{K}}^2 = \|\sum_P f_P(a)\Omega_P^{\mathbf{n}}\|_{\mathcal{K}}^2 \leq \sum_P \|f_P(a)\|_{L^2([0, \frac{2\pi}{L}])}^2 = \|f\|_{L^2(\mathbb{R})}^2 \leq (\text{const.})^2$. Finally, by linearity $(\mathbf{H} + \lambda_0)\Psi = \sum c_{l, \mathbf{n}, P_0} e^{ilLa}\Omega_{P_0}^{\mathbf{n}}$ has a solution in \mathcal{D} satisfying $\|\Psi\|_{\mathcal{K}}^2 \leq (\text{const.})^2 \sum |c_{l, \mathbf{n}, P_0}|^2 \leq (\text{const.})^2 \|\sum c_{l, \mathbf{n}, P_0} e^{ilLa}\Omega_{P_0}^{\mathbf{n}}\|_{\mathcal{K}}^2$, and since finite linear combinations of the $e^{ilLa}\Omega_{P_0}^{\mathbf{n}}$ constitute a dense set this proves density of the range of $\mathbf{H} + \lambda_0$. \square

6 The anomaly equation

In this section we derive the anomaly equation (1.2), which is the quantum analogue of the classical conservation law $\partial_\mu j_{\text{classical}}^{5, \mu} = 0$ discussed following (1.5). In fact we will work with the dressed fields, indicated by the use of boldface as in §5, and we will derive the fourier transformed version of (1.2):

$$\partial_t \mathbf{j}^1(m) + ik_m \mathbf{j}^0(m) = -\frac{1}{\pi} \mathbf{E}_m \quad (6.1)$$

where

$$\mathbf{E} = \sum_{m \in \mathbb{Z}} \mathbf{E}_m e^{ik_m x} = \mathbf{E}^{tr} + \mathbf{E}^{long} = \mathbf{E}^{tr} + \sum_{m \neq 0} \mathbf{E}_m^{long} e^{ik_m x}, \quad (6.2)$$

where, referring to §1, the operator corresponding to the longitudinal electric field has fourier components $\mathbf{E}_m^{long} = -e^2 \mathbf{j}^0(m)/(ik_m)$, as determined by the Gauss law. Therefore by (3.3) $\mathbf{E}_m = e^2 \Phi_m / \sqrt{\pi}$. (As remarked at the end of §5 the dressing operation acts trivially for the case $m = 0$, and so $\mathbf{E}_0 = E_0 = E^{tr}$ and $\Phi_0 = \Phi_0$ etc.) The anomaly equation (6.1) is therefore a consequence of (or, indeed, essentially equivalent to) the Heisenberg equation of motion $\dot{\mathbf{I}}_m = -\omega_m^2 \Phi_m$ since, by the definitions (3.3):

$$\partial_t \mathbf{j}^1(m) + ik_m \mathbf{j}^0(m) = \frac{1}{\sqrt{\pi}} (\dot{\mathbf{I}}_m + k_m^2 \Phi_m) = \frac{1}{\sqrt{\pi}} (-\frac{e^2}{\pi} \Phi_m) = -\frac{1}{\pi} \mathbf{E}_m. \quad (6.3)$$

This holds as an equality of operator valued distributions on \mathbb{R} .

We can see now that the anomaly occurs because we have enforced gauge invariance through the definitions (1.15)-(1.17) which were derived from Schwinger gauge invariant regularization. But as was made explicit in §2 the action of large gauge transformations on the positive energy representation Fock space does not respect the chiral phase invariance of the classical theory and Q^5 is not invariant under this group action. Thus enforcing gauge invariance of the Hamiltonian under this group action necessarily leads to a theory which does not respect the chiral symmetry - it can be traced back to the introduction of a non-invariant vacuum ("sea-level") in the positive energy representation. It is interesting to compare this with the picture of the spectral flow explained in [9] and [7, §6.6], in which the anomaly arises from the infinitely deep Dirac sea of filled levels (after regularization).

7 Concluding remarks and relation with previous work.

Since [11] there have been many treatments of the Schwinger model in the physics literature, see the bibliography in [9] or [13, 17] for a textbook treatment. The physics of the model and its massive generalization are discussed in [1]. Much of the previous rigorous mathematical work on two dimensional quantum electrodynamics, both the massless case of the Schwinger model studied here and the general massive case of $(QED)_2$,

has employed functional integration in Euclidean space-time - see [3, 4, 6, 15, 16]. These developments give a rigorous treatment of the Schwinger model, though with less transparency in respect of the points itemized below. In [5] the Hamiltonian approach is pursued within the Stückelberg indefinite metric formalism. In [2] the Coulomb gauge Hamiltonian for $(QED)_2$ is shown to be densely defined and a comparison with functional integral methods is made. However it should be said that the problem studied in [2, §2], restricted to the massless case, is strictly speaking a different model to that considered here: it is Coulomb gauge $(QED)_2$ on \mathbb{R} restricted by periodic boundary conditions to the circle, rather than $(QED)_2$ on the circle put into Coulomb gauge. (It is assumed that it is possible to eliminate the spatial component of the potential completely, which is not possible on the circle with a periodic gauge transformation but *is* possible with the whole real line as spatial domain. As a consequence the modular group \mathcal{M} does not appear in [2], and there is no description of the anomaly.)

In this paper we have provided a Hamiltonian formulation of the Schwinger model on the circle, incorporating the insights from [9] (regarding the modular group and the anomaly) into the positive energy representation. The advantage of this representation is that a rigorous definition of the Hamiltonian $H = H_0 + :H_{coul}:$ from (1.15)-(1.16) as a self-adjoint operator is then easily achieved through bosonization. Noteworthy conclusions are:

- There is a nontrivial action of $\mathcal{M} = \mathbb{Z}$, the modular group of large gauge transformations, on the non-interacting Fock vacuum Ω_0 given in 2 which maps Ω_0 to the unexcited states Ω_P . The Ω_P are eigenstates of the (unregularized) axial charge operator Q^5 with the following property: no excited fermionic state is occupied which has higher energy than an unoccupied one (amongst states with the correct sign of Q^5).
- It is necessary to take into account this action of \mathcal{M} in the definition of the Hamiltonian and currents in order to obtain the “correct” gauge invariant expressions in (1.15)-(1.17). These expressions, which are derived in [12] using Schwinger point-splitting regularization, contain terms which give rise to the anomaly in the chiral conservation law (1.2) and the mass of the fundamental boson described in §3.
- Even without “turning on” the Coulomb interaction the interaction between the fermions and the spatial component of the electromagnetic potential a destabilizes the gauge variant non-interacting vacuum Ω_0 , producing an interacting vacuum which is gauge invariant (as expressed by the twisted periodicity condition (2.29)). There is an explicit formula for the interacting vacuum as a linear combination of segments of a gaussian tensored with Ω_P , wrapped around the circle $S^1 = \mathbb{R}/(2\pi/L)$ in such a way as to satisfy (2.29) - see (4.7).
- The effect of turning on the Coulomb interaction is to transform the unexcited states Ω_P into dressed versions Ω_P , in terms of which the vacuum takes the same form - see (5.9).

It is to be hoped that the description of the Schwinger model and its anomaly offered here will, by virtue of its transparency, be helpful in further developments such as mass perturbation theory ([1, 4]), curved space-time and generalization to non-abelian gauge groups.

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References

- [1] S. Coleman, R. Jackiw and L. Susskind Charge shielding and quark confinement in the massive Schwinger model *Ann. Phys.* **93** 267-275 (1975).
- [2] J. Dimock $(QED)_2$ in the Coulomb gauge *Annales de l'Institut Henri Poincaré. Physique Theorique* **43**,2 167-179 (1985).
- [3] Fröhlich, J. Quantum sine-Gordon equation and quantum solitons in two space-time dimensions *Renormalization theory (Proc. NATO Advanced Study Inst., Erice, 1975). NATO Advanced Study Inst. Series C: Math. and Phys. Sci., Vol. 23*, 371-414. Reidel, Dordrecht, 1976.
- [4] Fröhlich, J. and Seiler, E. The massive Thirring-Schwinger model $(QED)_2$: convergence of perturbation theory and particle structure *Helv. Phys. Acta* **49**, 6 889-924 (1976).

- [5] Ito, K.R. Construction of two dimensional quantum electrodynamics *J. Math. Phys.* **21**,6 1473-1494 (1980).
- [6] Ito, K.R. Construction of euclidean $(QED)_2$ via lattice gauge theory *Comm. Math. Phys.* **83**, 537-561 (1982).
- [7] R. Jackiw Topological investigations of quantized gauge theories, reprinted in *Current algebra and anomalies*, by Treiman, Jackiw, Zumino and Witten, PUP, Princeton, New Jersey, 1985.
- [8] D. Mattis and E. Lieb Exact solution of a many-fermion system and its associated boson field *J. Math. Phys.* **6** 304-312 (1965).
- [9] N.S. Manton The Schwinger model and its axial anomaly *Ann. Phys.* **159** 220-251 (1985).
- [10] M. Reed and B. Simon *Methods of modern mathematical physics. Vols I and II*, Academic Press, New York, 1980.
- [11] J. Schwinger Gauge Invariance and Mass II *Physical Review* **128**, no. 5, 2425 (1962).
- [12] D.M.A. Stuart Notes on the Schwinger model: regularization and gauge invariance. Online notes available as arXiv:1206.0878
- [13] A. Tsvelik *Quantum field theory in condensed matter physics* CUP, Cambridge, 2003
- [14] D.A. Uhlenbrock Fermions and associated bosons of one-dimensional model *Comm. Math. Phys.* **4**, 64-76 (1967).
- [15] D. Weingarten and J. Challifour Continuum limit of $(QED)_2$ on a lattice *Ann. Phys.* **123** 61-101 (1979).
- [16] D. Weingarten Continuum limit of $(QED)_2$ on a lattice II *Ann. Phys.* **126** 154-175 (1980).
- [17] J. Zinn-Justin *Quantum field theory and critical phenomena* OUP, Oxford, 2002